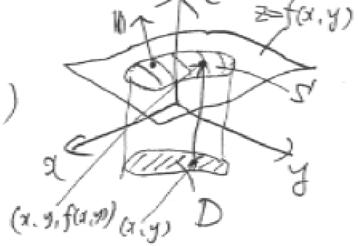
例題2 (P62)　(ただし $n_z>0$ とする)



(1) $\mathbf{r}=\mathbf{r}(x,y)=(x,\ y,\ f(x,y))$　$((x,y)\in D)$

より $\dfrac{\partial \mathbf{r}}{\partial x}=\left(1,0,\dfrac{\partial f}{\partial x}\right),\ \dfrac{\partial \mathbf{r}}{\partial y}=\left(0,1,\dfrac{\partial f}{\partial y}\right)$

$\dfrac{\partial \mathbf{r}}{\partial x}\times\dfrac{\partial \mathbf{r}}{\partial y}=\left(-\dfrac{\partial f}{\partial x},-\dfrac{\partial f}{\partial y},1\right)$　$\mathbf{n}$と同じ向き ($n_z>0$)

$\left|\dfrac{\partial \mathbf{r}}{\partial x}\times\dfrac{\partial \mathbf{r}}{\partial y}\right|=\sqrt{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}$

よって

$$\mathbf{n}=+\frac{\frac{\partial \mathbf{r}}{\partial x}\times\frac{\partial \mathbf{r}}{\partial y}}{\left|\frac{\partial \mathbf{r}}{\partial x}\times\frac{\partial \mathbf{r}}{\partial y}\right|}=\frac{\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\right)}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}\quad —①$$

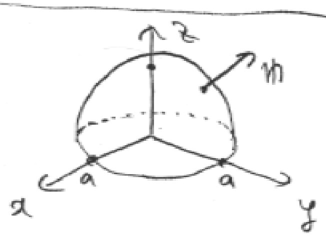
(2) (1)より

$$\begin{cases} dS=\left|\dfrac{\partial \mathbf{r}}{\partial x}\times\dfrac{\partial \mathbf{r}}{\partial y}\right|dx\,dy=\sqrt{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}\,dx\,dy\quad —② \\ \mathbf{n}\,dS=①\times②=\left(-\dfrac{\partial f}{\partial x},-\dfrac{\partial f}{\partial y},1\right)dx\,dy\quad —③ \end{cases}$$

$$\therefore \int_S \mathbf{A}\cdot\underbrace{\mathbf{n}\,dS}_{③}=\iint_D \mathbf{A}\cdot\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\right)dx\,dy$$

$$=\iint_D\left(-A_x\frac{\partial f}{\partial x}-A_y\frac{\partial f}{\partial y}+A_z\right)dx\,dy$$

例題3 (P63)　(ただし $n_z\geq 0$ とする)



S: $x^2+y^2+z^2=a^2,\ z\geq 0$ (北半球)

$\mathbf{A}=(x,y,-2z)$ $\Rightarrow$ $\int_S \mathbf{A}\cdot\mathbf{n}\,dS=?$

$\mathbf{r}=(a\cos\theta\cos\phi,\ a\cos\theta\sin\phi,\ a\sin\theta)$ —①　$\left(0\leq\theta\leq\frac{\pi}{2},\ 0\leq\phi<2\pi\right)$ $\Rightarrow D$

北半球

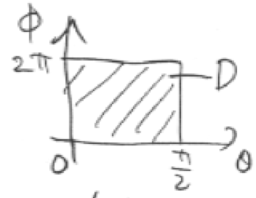
でやってみる。

$\dfrac{\partial \mathbf{r}}{\partial\theta}\times\dfrac{\partial \mathbf{r}}{\partial\phi}=(-a\cos\theta)\mathbf{r}$ (← $\mathbf{r}$と逆向き，よって$\mathbf{n}$と逆向き)

$\therefore\ \mathbf{n}=-\dfrac{\partial \mathbf{r}}{\partial\theta}\times\dfrac{\partial \mathbf{r}}{\partial\phi}\Big/\left|\dfrac{\partial \mathbf{r}}{\partial\theta}\times\dfrac{\partial \mathbf{r}}{\partial\phi}\right|$ —②

$$\mathbf{n}\,dS=②\times\underbrace{\left|\frac{\partial \mathbf{r}}{\partial\theta}\times\frac{\partial \mathbf{r}}{\partial\phi}\right|d\theta\,d\phi}_{dS}=-\frac{\partial \mathbf{r}}{\partial\theta}\times\frac{\partial \mathbf{r}}{\partial\phi}\,d\theta\,d\phi$$

$$=(+a\cos\theta\,\mathbf{r})\,d\theta\,d\phi\quad —③$$



θ,ϕ の範囲は 右図の長方形 D

$$\therefore \int_S \mathbf{A}\cdot\mathbf{n}\,dS=\iint_D (x,y,-2z)\cdot\{a\cos\theta\underbrace{(x,y,z)}_{\mathbf{r}}\}\,d\theta\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(x^2+y^2-2z^2)\,a\cos\theta\,d\theta\,d\phi$$

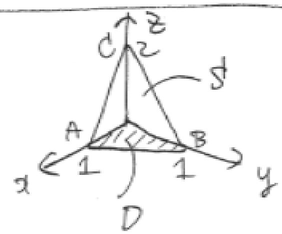
$\Big(x^2+y^2-2z^2\underset{①}{=}a^2\cos^2\theta\cos^2\phi+a^2\cos^2\theta\sin^2\phi-2a^2\sin^2\theta$

$=a^2(1-3\sin^2\theta)\Big)$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}a^3(1-3\sin^2\theta)\cos\theta\,d\theta\,d\phi\quad(t=\sin\theta と置換)$$

$$=\int_0^{2\pi}\int_0^1 a^3(1-3t^2)\,dt\,d\phi=\int_0^{2\pi}a^3\left[t-t^3\right]_{t=0}^{t=1}d\phi=0$$

P69 問題3.4の1　($n_z>0$)



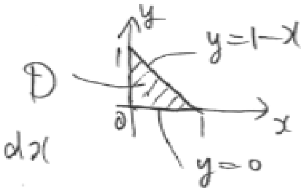
Hint:

S: $z=2-2x-2y$

D: $0\leq y\leq 1-x$

$0\leq x\leq 1$

なので 例題2と同様。



$$\iint_D ▨\,dx\,dy=\int_0^1\int_0^{1-x}▨\,dy\,dx$$

(又は $=\int_0^1\int_0^{1-y}▨\,dx\,dy$)

教科書